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Collective Modes and Coulomb Effects in Superconductivity

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Abstract

The effect of the Coulomb interaction in superconductivity is investigated by means of the self-consistent method employed in the boson formulation of superconductivity. The charge and current density of both quasi fermion and boson parts are calculated. It is found that the charge and current density in the ground state are dominated by the boson part. Also, our study seems to indicate that the Goldstone theorem is still valid, even in the presence of long range forces in the sense that a boson condensation is permitted. This massless particle recovers the gauge invariance and is responsible for the dynamical rearrangement of symmetry.

1) Introduction

It is well known in the theory of superconductivity that the presence of collective modes is essential in order to have a theory which is implicitly gauge invariant. In the original BCS formulation the derivation of the Meissner effect was not gauge invariant; this feature was soon understood as due to an unsatisfactory level of the perturbation methods used. Several calculations[1] performed in approximations of higher order than a simple Hartree-Fock approximation showed how at this level the gauge invariance of the theory is restored. At the same time, in higher order approximations, collective states which have been interpreted as bound states of quasi-electrons with opposite momentum and spin appear. These collective modes, which behave as boson fields, restore the gauge invariance and the conservation of the current. In this sense the presence of these modes is essential in the construction of a theory of superconductivity which is naturally gauge invariant at every stage of the calculation.

This situation has motivated a different approach, the boson method in superconductivity, which has been developed in a series of articles[2,3,4]. The attitude of this approach is based on the fact that the role played by the boson field should be investigated at every level of the theory. In the course of our study we have found that the inclusion of the boson field besides recovering the gauge invariance, enables us to formulate the theory in a systematic and unified form which

offers several practical advantages when applied to particular problems, such as the appearance of flux lines[5] in the mixed state of type-II superconductors and Josephson effects[6]. Indeed the boson method turns out to be of practical applicability in those regimes where the Gor'kov equations cannot be reduced in the form of differential equations (Ginzburg-Landau equations and their generalizations)[7]. In preceding articles[3,4] the charge and current density has been assumed to be principally described by the boson field; the quasi-fermion contribution has been considered negligible. One of the results of the present article shows that this assumption is correct.

A study of these collective modes is also important from a more fundamental point of view in the phenomenon known as spontaneous breakdown of symmetries. There exists in nature a variety of many-particle systems which share in common the feature that the observed behavior does not present all the symmetry properties possessed by their underlying Hamiltonians. In general these systems have a ground state which presents a systematic structure and is asymmetric with respect to the symmetry transformation. In the case of superfluid systems (superfluid Helium and superconductors) the Hamiltonian is invariant under gauge transformations, but this symmetry is phenomenologically violated. It is a fact that a collective oscillation whose energy spectrum vanishes in the limit of long wave lengths is always associated with the presence of an asymmetry in the ground state. This situation has been proved

in complete generality in the well known Goldstone theorem[8]. It is also known that this theorem does not provide a complete description of the situation since the fundamental question of what happened to the original symmetry remains unanswered. Since the invariance of the theory must be kept at every stage, it may happen that owing to the highly non linear character of the interactions the symmetry takes a different form when the theory is expressed in terms of quasi-particles (incoming fields in the terminology of quantum field theory). The conservation laws will still hold at the level of the physical fields (observed particles), only they will show different forms according to the new shape taken by the symmetry. This phenomenon has been denoted by the name of dynamical rearrangement of symmetries[9]. In the process of rearrangement of symmetry the Goldstone particle plays a principal role; in the course of numerous calculations performed on different models it has always been found that the Goldstone particles rearrange the symmetry[10].

It is now a fact that the Goldstone theorem does not appear consistent with the experimental situation in superconductivity; this circumstance has led one to expect that the Goldstone theorem may break down when long range interaction of the Coulomb type are present. In particular let us consider the case of superfluid liquids and superconductors; below a certain critical temperature both systems exhibit the phenomenon of condensation, which implies an asymmetric ground state and spontaneous breakdown of symmetry. It is phenomenologically

observed that in superfluid liquids there are collective modes whose energy goes to zero in the long wave length limit, while in superconducting systems such modes possess a finite energy gap. The only sensible difference which exists between these two systems is the presence of a Coulomb interaction among the particles. The disappearance of the Goldstone particle and the presence of long range interactions were theoretically related by Anderson[1]. He found that a neutral system of fermions with an attractive short range interaction exhibits a collective mode whose energy vanishes in the limit of long wave lengths; when a Coulomb potential is introduced, these collective excitations acquire a finite mass which coincides with the value of plasmons which are usually observed in many-electron systems.

Other examples of a failure of the Goldstone theorem have been studied in relativistic models containing gauge fields[11,12].

However all these theoretical investigations do not provide an answer to the fundamental question: how is the symmetry rearranged without the Goldstone particle?

In the present article we consider a superconducting system taking into account the effects of the Coulomb potential. We are particularly interested in studying how the presence of long range correlations influences the collective modes. This problem is of interest at two levels. On one hand it is important in the framework of the boson method since the central point[4] of this method is the introduction of some invariant transformations expressed solely in terms of the boson field.

By applying these transformations we can describe space and time dependent situations[5,6]. On the other hand this problem is of general interest in the context of the spontaneous breakdown of symmetries. In particular we wish to understand the effect of long range forces on the Goldstone theorem. Does the Goldstone particle completely disappear from the theory? How is the symmetry rearranged at the level of the physical fields? These questions have been left unanswered in previous treatments, but in our opinion they play a basic role. In this article we examine these problems and our study gives a very clear answer to these questions.

By applying a self-consistent calculation[13] in the pair approximation to the well-known Hamiltonian for electrons in superconductors we express the operator form of most of the observable quantities in terms of quasi-electron and boson fields. We prefer to express the theory in terms of these operator forms instead of the usual approach based on the Green functions because these operator expressions permit us to perform those invariant operator transformations (boson transformations) which represent the cornerstone of the boson method. Furthermore these operator forms permit us to investigate the problem of the rearrangement of the symmetry.

In section 2 we derive the equations which fix the coefficients of the dynamical map. These equations are solved in sections 3 and 4 where explicit expressions of the current and charge density in terms of the physical fields are derived.

Finally in section 5 we discuss the results obtained in connection with the Goldstone theorem and the dynamical rearrangement of the symmetry. We obtain an equation for the boson field which differs in a sensible way from other equations obtained in different approaches[1] in the sense that it permits not only excitations at plasma level but also condensation in the ground state. This condensation carries the gauge transformation and is responsible for the invariance of the theory at the level of the physical fields. The presence of this condensation is also important in practical applications[5,6] since it permits us to treat time- and space-dependent phenomena in superconductivity avoiding the different problem of solving the Gor'kov equations.

In closing this section we would like to mention that the problem of Coulomb effects has been already investigated in Ref. 3; however, the treatment is not completely satisfactory since the effect of the quasi-fermion contribution to the charge density was completely neglected. In this article we apply the self-consistent method to the BCS Hamiltonian including the Coulomb potential; we find that the expressions of the quasi-fermion contributions to the charge and current density are strongly suppressed by the Coulomb potential with respect to the neutral case. As a consequence of this result the boson transformation, introduced in Ref. 4, turns out to coincide with the original gauge transformation for all the frequencies of physical interest.

2) Dynamical Map

A superconducting system in the absence of magnetic fields is described by the well-known Hamiltonian:

$$H = H_o + H_1 + H_c \tag{2.1}$$

$$H_o = \Sigma_s \int d^3x \psi_s^\dagger(x)\, \epsilon(\vec{\nabla}^2)\, \psi_s(x) \tag{2.2}$$

$$H_1 = \lambda \int d^3x \psi_\uparrow^\dagger(x)\, \psi_\downarrow^\dagger(x)\, \psi_\downarrow(x)\, \psi_\uparrow(x) \tag{2.3}$$

$$H_c = \frac{e^2}{2} \int d^3x \int d^3y\, \frac{\rho(x)\,\rho(y)}{|\vec{x}-\vec{y}|} \tag{2.4}$$

where

$$\epsilon(\vec{\nabla}^2) = -\frac{1}{2m}(\vec{\nabla}^2 + k_F^2) \tag{2.5}$$

$$\rho(x) = \psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x) \tag{2.6}$$

and $s = \uparrow,\downarrow$. Following the methods of Refs. 13 and 2 we shall assume that the equilibrium state of a superconductor can be described in terms of a set of free fields $\{\Phi_{\uparrow,\downarrow}, B\}$; $\Phi_{\uparrow,\downarrow}$ corresponding to the quasi-electron fields; B corresponds to a collective field. It should be noted that the existence and the properties of these fields are not assumed; they are the results of the dynamics of the system described by the Hamiltonian (2.1) and will be determined in the course of a self-consistent calculation. Introducing the corresponding set of creation and annihilation

operators $\{\alpha^{\dagger}_{k\uparrow,\downarrow}(t),\alpha_{k\uparrow,\downarrow}(t)\}$ and $\{B^{\dagger}_{k}(t),B_{k}(t)\}$ we shall construct a Fock space as a realization of the canonical commutation relations:

$$[\alpha_{ks},\alpha^{\dagger}_{pr}]_{+} = \delta_{rs}\delta(\vec{k}-\vec{p}) \tag{2.7}$$

$$[B_{k},B^{\dagger}_{p}]_{-} = \delta(\vec{k}-\vec{p}) \tag{2.8}$$

We shall not introduce any other Fock space; every other operator present in the theory should be expressed in terms of the free operators. Thus, introducing the annihilation and creation operators $a_{k\uparrow,\downarrow},a^{\dagger}_{k\uparrow,\downarrow}$ of the fields $\psi_{\uparrow,\downarrow}$:

$$\psi_{\uparrow,\downarrow}(x) = \Sigma_{k}\, e^{i\vec{k}\cdot\vec{x}}a_{k\uparrow,\downarrow}(t) \tag{2.9}$$

we shall express them as a linear combination of normal products of the free operators:

$$a_{k\uparrow}(t) = \cos\theta_{k}\bar{\alpha}_{k\uparrow}(t) - \sin\theta_{k}\bar{\alpha}^{\dagger}_{-k\downarrow}(t) \tag{2.10}$$

here $\bar{\alpha}$ and $\bar{\alpha}^{\dagger}$ are intermediate operators expressed in terms of the physical fields. Since in the following we are going to use only bylinear products of such operators we shall give here only the expressions for these quantities:

$$\bar{\alpha}_{-k+\ell\downarrow}(t)\bar{\alpha}_{k\uparrow}(t) = \Sigma_{p}[f^{(1)}_{k}(p,\ell)\alpha_{\ell-p\uparrow}(t)\alpha_{p\downarrow}(t) - f^{(2)*}_{k}(p,\ell)\alpha^{\dagger}_{-\ell+p\downarrow}(t)\alpha^{\dagger}_{-p\uparrow}(t)] \tag{2.11}$$

$$+ g^{(1)}_{k}(\ell)B_{\ell}(t) - g^{(2)*}_{k}(\ell)B_{-\ell}(t) + \cdots\cdots$$

and similar expressions for the other combinations.

Such an expansion (2.10) of a Heisenberg field in terms of free fields has been denoted[13] by the name of dynamical map. The dots in (2.11) stand for higher-order normal products. Let us note that the terms of the expansion (2.11) are linearly independent of each other; therefore the problem of convergence does not exist. In (2.10) the time dependence of $a_{k\uparrow,\downarrow}$ comes only through the time dependence of the operators $\alpha_{k\uparrow,\ell}$ and B_k:

$$\alpha_{k\uparrow,\downarrow}(t) = \alpha_{k\uparrow,\downarrow} e^{-iE_k t} \tag{2.12}$$

$$B_k(t) = B_k e^{-i\omega_k t} \tag{2.13}$$

The quantities θ_k, $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$, E_k and ω_k are to be determined by requiring that ψ satisfies the canonical equations of motion which correspond to the Hamiltonian (2.1).

Introducing

$$H_s = \Delta \int d^3x [\psi_\uparrow(x)\psi_\downarrow(x) + \psi_\downarrow^\dagger(x)\psi_\uparrow^\dagger(x)] \tag{2.14}$$

where Δ, the order parameter, is defined as

$$\Delta = \lambda \langle 0 | \psi_\uparrow(x)\psi_\downarrow(x) | 0 \rangle \tag{2.15}$$

and rewriting the Hamiltonian (2.1) as

$$H = (H_o + H_s) + (H_1 + H_c - H_s) = H_o' + H_{int} \tag{2.16}$$

the coefficients in (2.10) are fixed by requiring the diagonalization of H_o'. It is well known that

$$\cos 2\theta_k = \varepsilon_k / E_k \qquad \sin 2\theta_k = -\Delta / E_k \tag{2.17}$$

where

$$\varepsilon_k = \frac{1}{2m}(\vec{k}^2 - k_F^2) \tag{2.18}$$

$$E_k = [\varepsilon_k^2 + \Delta^2]^{1/2} \tag{2.19}$$

Introducing the notation

$$\hat{\psi} = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^\dagger \end{pmatrix} \tag{2.20}$$

we shall define the following Bethe-Salpeter amplitudes

$$F(x,y;p,q) = \langle 0|T[\hat{\psi}(x)\hat{\psi}^\dagger(y)]|\alpha_{p\downarrow}\alpha_{q\uparrow}\rangle \tag{2.21}$$

$$G(x,y;\ell) = \langle 0|T[\hat{\psi}(x)\hat{\psi}^\dagger(y)]|B_\ell\rangle \tag{2.22}$$

where $|\alpha\alpha\rangle$ denotes a state of two quasi-electrons: $|\alpha_{p\downarrow}\alpha_{q\uparrow}\rangle = \alpha_{p\downarrow}^\dagger\alpha_{q\uparrow}^\dagger|0\rangle$ and $|B\rangle$ a state of single boson $|B_\ell\rangle = B_\ell^\dagger|0\rangle$. By requiring that the coefficients of the dynamical map have the form of retarded functions, the quantities $f^{(1)}$, $f^{(2)}$, $g^{(1)}$ and $g^{(2)}$ can be expressed as[2]:

$$f_k^{(1)}(p,\ell) = \frac{\lambda}{E_k+E_{k-\ell}-E-i\varepsilon}\,[\cos\theta_k\cos\theta_{k-\ell}F^{(1)}(p,\ell)+\sin\theta_k\sin\theta_{k-\ell}F^{(2)}(p,\ell) - \tfrac{1}{2}(\sin\theta_k\cos\theta_{k-\ell}+\cos\theta_k\sin\theta_{k-\ell})f(p,\ell)] \tag{2.23}$$

$$f_k^{(2)}(p,\ell) = \frac{-\lambda}{E_k+E_{k-\ell}+E+i\varepsilon}\,[\sin\theta_k\sin\theta_{k-\ell}F^{(1)}(p,\ell)+\cos\theta_k\cos\theta_{k-\ell}F^{(2)}(p,\ell) + \tfrac{1}{2}(\sin\theta_k\cos\theta_{k-\ell}+\cos\theta_k\sin\theta_{k-\ell})f(p,\ell)] \tag{2.24}$$

$$g_k^{(1)}(\ell) = \frac{\lambda}{E_k+E_{k-\ell}-\omega_\ell-i\varepsilon}\,[\cos\theta_k\cos\theta_{k-\ell}G^{(1)}(\ell)+\sin\theta_k\sin\theta_{k-\ell}G^{(2)}(\ell) \tag{2.25}$$
$$-\frac{1}{2}(\sin\theta_k\cos\theta_{k-\ell}+\cos\theta_k\sin\theta_{k-\ell})g(\ell)]$$

$$g_k^{(2)}(\ell) = \frac{-\lambda}{E_k+E_{k-\ell}+\omega_\ell+i\varepsilon}\,[\sin\theta_k\sin\theta_{k-\ell}G^{(1)}(\ell)+\cos\theta_k\cos\theta_{k-\ell}G^{(2)}(\ell) \tag{2.26}$$
$$+\frac{1}{2}(\sin\theta_k\cos\theta_{k-\ell}+\cos\theta_k\sin\theta_{k-\ell})g(\ell)]$$

In (2.23) and (2.24) $E = E_p+E_{p-\ell}$. The functions $F^{(1)}$, $F^{(2)}$, f, $G^{(1)}$, $G^{(2)}$ and g are related to the B-S reduced amplitudes

$$F(x,x;p,\ell-p) = e^{i(\vec{x}\cdot\vec{\ell}-Et)}F(p,\ell) \tag{2.27}$$

$$G(x,x;\ell) = e^{i(\vec{x}\cdot\vec{\ell}-\omega_\ell t)}G(\ell) \tag{2.28}$$

by

$$F^{(1)} = F_{12} \quad F^{(2)} = -F_{21} \quad f = F_{22}-F_{11} \tag{2.29}$$

$$G^{(1)} = G_{12} \quad G^{(2)} = -G_{21} \quad g = G_{22}-G_{11} \tag{2.30}$$

The expressions (2.23)-(2.26) are valid in the pair approximation in which we take into account only the Feynman diagrams which conserve the number of pairs. This approximation will be used consistently in all of the remaining part of the present work. It should be noted that in this approximation the terms denoted by dots in (2.11) do not contribute.

In the absence of a Coulomb potential the Bethe-Salpeter equations for the amplitudes (2.21) and (2.22) have been derived in Ref. 2. When the Coulomb interaction is taken

into account these equations take the form:

$$D(x,y) = D_o(x,y) - i \int d^4z S(x-z) F(z) S(z-y) \tag{2.31}$$

where

$$D(x,y) = \langle 0 | T[\hat{\psi}(x) \hat{\psi}^{\dagger}(y)] | \ell \rangle \tag{2.32}$$

$|\ell\rangle$ being any physical state different from the vacuum, i.e. $\langle 0|\ell\rangle = 0$. D_o is the B-S amplitude in the absence of interactions and $S(x)$ satisfies the equation

$$[\varepsilon(\vec{\nabla}^2)\tau_3 + \frac{1}{i}\frac{\partial}{\partial t} - \Delta\tau_1]S(x) = I\delta(x) \tag{2.33}$$

where τ_1 and τ_3 are the Pauli matrices. The function $F(x)$ in (2.31) is defined as

$$F_{11}(x) = -\lambda\{D_{22}(x,x) + \frac{1}{2} q(\vec{\ell}) [D_{22}(x,x) - D_{11}(x,x)]\}$$

$$F_{12}(x) = \lambda D_{12}(x,x) \qquad F_{21}(x) = \lambda D_{21}(x,x) \tag{2.34}$$

$$F_{22}(x) = -\lambda\{D_{11}(x,x) + \frac{1}{2} q(\vec{\ell}) [D_{11}(x,x) - D_{22}(x,x)]\}$$

$q(\vec{\ell})$ is given by

$$q(\vec{\ell}) = \frac{2e^2}{\lambda} \int d^3x \frac{1}{|\vec{x}|} e^{i\vec{\ell}\cdot\vec{x} - \varepsilon|\vec{x}|} = \frac{8\pi e^2}{\lambda(\varepsilon^2 + \vec{\ell}^2)} \tag{2.35}$$

where the infinitesimal quantity ε has been introduced in order to define integrations which contain Coulomb potential. From (2.31) it is straightforward to derive the equations for the B-S reduced amplitudes (2.27) and (2.28):

$$
\begin{cases}
F^{(1)}(p,\ell) = \cos\theta_p \cos\theta_{p-\ell} + Q^{11}(\vec{\ell},E)F^{(1)}(p,\ell) + Q^{33}(\vec{\ell},E)F^{(2)}(p,\ell \\
\qquad - Q^{13}(\vec{\ell},E)[1+q(\vec{\ell})]f(p,\ell) \\
F^{(2)}(p,\ell) = \sin\theta_p \sin\theta_{p-\ell} + Q^{33}(\vec{\ell},E)F^{(1)}(p,\ell) + Q^{22}(\vec{\ell},E)F^{(2)}(p,\ell \\
\qquad - Q^{23}(\vec{\ell},E)[1+q(\vec{\ell})]f(p,\ell) \\
f(p,\ell) = \cos\theta_p \sin\theta_{p-\ell} + \sin\theta_p \cos\theta_{p-\ell} - 2Q^{13}(\vec{\ell},E)F^{(1)}(p,\ell) \\
\qquad - 2Q^{23}(\vec{\ell},E)F^{(2)}(p,\ell) + [Q^{12}(\vec{\ell},E)+Q^{33}(\vec{\ell},E)][1+q(\vec{\ell})]f(p,
\end{cases}
\tag{2.36}
$$

for the quasi-fermion B-S amplitude, and

$$
\begin{cases}
G^{(1)}(\ell) = Q^{11}(\vec{\ell},\omega_\ell)G^{(1)}(\ell) + Q^{33}(\vec{\ell},\omega_\ell)G^{(2)}(\ell) \\
\qquad - Q^{13}(\vec{\ell},\omega_\ell)[1+q(\vec{\ell})]g(\ell) \\
G^{(2)}(\ell) = Q^{33}(\vec{\ell},\omega_\ell)G^{(1)}(\ell) + Q^{22}(\vec{\ell},\omega_\ell)G^{(2)}(\ell) \\
\qquad - Q^{23}(\vec{\ell},\omega_\ell)[1+q(\vec{\ell})]g(\ell) \\
g(\ell) = -2Q^{13}(\vec{\ell},\omega_\ell)G^{(1)}(\ell) - 2Q^{23}(\vec{\ell},\omega_\ell)G^{(2)}(\ell) \\
\qquad + [Q^{12}(\vec{\ell},\omega_\ell)+Q^{33}(\vec{\ell},\omega_\ell)][1+q(\vec{\ell})]g(\ell)
\end{cases}
\tag{2.37}
$$

for the boson B-S amplitude.

The coefficients Q^{ij} are defined in the Appendix A. It is interesting to note that the effect of the Coulomb potential can be conglomerated only as a factor $q(\vec{\ell})$ multiplying the amplitudes $f(p,\ell)$ and $g(\ell)$. We shall postpone till the later sections the problem of solving the two set of equations (2.3 and (2.37). Here we shall limit ourselves to mention that th equations should be solved taking into account the self-consi

condition

$$1 + \lambda \int \frac{d^3k}{(2\pi)^3} \frac{E_k + E_{k-\ell}}{4E_k E_{k-\ell}} = 0 \qquad (2.38)$$

This equation is the generalized gap equation proposed and justified in Ref. 2. By means of this equation the following important relation can be derived:

$$1 - Q^+(\vec{\ell},E) - Q^{33}(\vec{\ell},E) = (E^2 - \bar{\omega}_\ell^2) R(\vec{\ell},E) \qquad (2.39)$$

where Q^+ is defined in the Appendix A and

$$R(\vec{\ell},E) = \lambda \int \frac{d^3k}{(2\pi)^3} \frac{E_{k+\frac{1}{2}\ell} + E_{k-\frac{1}{2}\ell}}{4E_{k+\frac{1}{2}\ell} E_{k-\frac{1}{2}\ell}} \frac{1}{[(E_{k+\frac{1}{2}\ell} + E_{k-\frac{1}{2}\ell})^2 - E^2]} \qquad (2.40)$$

$$\bar{\omega}_\ell^{-2} = \frac{\lambda}{R(\vec{\ell},E)} \int \frac{d^3k}{(2\pi)^3} \frac{E_{k+\frac{1}{2}\ell} + E_{k-\frac{1}{2}\ell}}{4E_{k+\frac{1}{2}\ell} E_{k-\frac{1}{2}\ell}} \frac{(\varepsilon_{k+\frac{1}{2}\ell} - \varepsilon_{k-\frac{1}{2}\ell})^2}{[(E_{k+\frac{1}{2}\ell} + E_{k-\frac{1}{2}\ell})^2 - E^2]} \qquad (2.41)$$

3) Calculation of the Fermion Current

a) Charge Density

Let us consider the charge density

$$\rho(x) = \psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x) \qquad (3.1)$$

and let us study the contribution due to the quasi-fermions. From the definitions (2.21), (2.27) and (2.29) we have

$$\langle 0 | \rho(x) | \alpha_{p\downarrow}\alpha_{\ell-p\uparrow} \rangle = f(x;p,\ell) \qquad (3.2)$$

so that in the pair channel:

$$\rho^F(x) = \int\frac{d^3\ell}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\, e^{i[\vec{x}\cdot\vec{\ell}-t(E_p+E_{p-\ell})]}\, f(p,\ell)\alpha_{\ell-p\uparrow}\alpha_{p\downarrow}+h.c. \tag{3.3}$$

Solving the system of linear equations (2.36) $f(p,\ell)$ is found to be

$$f(p,\ell) = \frac{1}{W(\vec{\ell},E)}[\sin(\theta_p+\theta_{p-\ell})X_3(\vec{\ell},E)-2\cos(\theta_p+\theta_{p-\ell})X_2(\vec{\ell},E)$$

$$- 2E\cos(\theta_p-\theta_{p-\ell})Z_2(\vec{\ell},E)] \tag{3.4}$$

where $E = E_p+E_{p-\ell}$ and

$$W(\vec{\ell},E) = \{R(\vec{\ell},E)Y(\vec{\ell},E)-Z(\vec{\ell},E)+q(\vec{\ell})[Y'(\vec{\ell},E)-Z'(\vec{\ell},E)]\}[E^2-\bar{\omega}_\ell^2 K(\vec{\ell},E)] \tag{3.5}$$

The quantities that appear in (3.4) and (3.5) are defined in Appendix B. From the expressions (B.12), (B.13) and (B.15) $W(\vec{\ell},E)$ takes the form

$$W(\vec{\ell},E) = R^2(\vec{\ell},E)(E^2-\bar{\omega}_\ell^2-4\Delta^2)[E^2-\omega_p^2(\vec{\ell})] \tag{3.6}$$

where $\omega_p^2(\vec{\ell})$ is defined by

$$\omega_p^2(\vec{\ell}) = \bar{\omega}_\ell^2[1+4\Delta^2R+4\Delta^2Rq(\vec{\ell})] \tag{3.7}$$

Substituting (3.6) in (3.4) and using the expressions (B.10), (B.11) and (B.14), we obtain the following expression for $f(p,\ell)$:

$$f(p,\ell) = \frac{\sin(\theta_p+\theta_{p-\ell})\,(E^2-\bar{\omega}_\ell^2)+2\Delta E\cos(\theta_p-\theta_{p-\ell})}{E^2-\omega_p^2(\vec{\ell})} \tag{3.8}$$

Let us now denote the quasi-fermion field by

$$\hat{\phi} = \begin{pmatrix} \phi_\uparrow \\ \phi_\downarrow^\dagger \end{pmatrix} \tag{3.9}$$

which satisfies the equation of motion

$$[\varepsilon(\vec{\nabla}^2)\tau_3 + \frac{1}{i}\frac{\partial}{\partial t} - \Delta\tau_1]\hat{\phi}(x) = 0 \tag{3.10}$$

Then let us define the following operators

$$\vec{j}(0) = \frac{1}{2mi}[\phi_\uparrow^\dagger\vec{\nabla}\phi_\uparrow-(\vec{\nabla}\phi_\uparrow^\dagger)\phi_\uparrow+\phi_\downarrow^\dagger\vec{\nabla}\phi_\downarrow-(\vec{\nabla}\phi_\downarrow^\dagger)\phi_\downarrow] \tag{3.11}$$

$$\rho^{(0)} = \phi_\uparrow^\dagger\phi_\uparrow+\phi_\downarrow^\dagger\phi_\downarrow \tag{3.12}$$

The expressions (3.11) and (3.12) correspond to the current and charge density in the usual Hartree-Fock approximation. In the self-consistent method they correspond to the current and charge density calculated to the lowest order approximation; that is to the first order in the dynamical map. It is well known that at such order the current is not conserved, but

$$\partial_\mu j_\mu^{(0)} = \vec{\nabla}\cdot\vec{j}^{(0)} + \frac{\partial}{\partial t}\rho^{(0)} = 2i\Delta(\phi_\uparrow\phi_\downarrow-\phi_\downarrow^\dagger\phi_\uparrow^\dagger) \tag{3.13}$$

By diagonalizing the equation (3.11) we find the expression of $\phi_\uparrow$ and $\phi_\downarrow$ in terms of free fields:

$$\phi_{\uparrow}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{x}\cdot\vec{k}} [\cos\theta_k \alpha_{k\uparrow}(t) - \sin\theta_k \alpha^{\dagger}_{-k\downarrow}(t)] \qquad (3.14)$$

$$\phi_{\downarrow}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{x}\cdot\vec{k}} [\cos\theta_k \alpha_{k\downarrow}(t) + \sin\theta_k \alpha^{\dagger}_{-k\uparrow}(t)] \qquad (3.15)$$

Substituting the expression (3.8) in (3.3) and introducing the Fourier transforms of (3.11) and (3.13) we can easily see that the Fourier component of the fermion contribution to the charge density has the form:

$$\rho^F(\vec{\ell},t) = \frac{1}{\frac{\partial^2}{\partial t^2} + \omega_p^2(\vec{\ell})} \left[\left(\frac{\partial^2}{\partial t^2} + \bar{\omega}_\ell^2 \right) \rho^{(0)}(\vec{\ell},t) - \frac{\partial}{\partial t} (\partial_\mu j^{(0)}_\mu)(\vec{\ell},t) \right] \qquad (3.16)$$

By noticing that $\bar{\omega}_\ell$ vanishes at $\vec{\ell} = 0$ and that

$$\left[\frac{\partial}{\partial t} \rho^{(0)} - \partial_\mu j^{(0)}_\mu \right]_{\ell=0} = 0 \qquad (3.17)$$

we obtain from (3.16) the important result

$$\int d^3x \, \rho^F(x) = 0 \qquad (3.18)$$

b) Current Density

The electron current density is defined by

$$\vec{j}(x) = \frac{1}{2mi} [\psi^{\dagger}_{\uparrow} \vec{\nabla} \psi_{\uparrow} - (\vec{\nabla}\psi^{\dagger}_{\uparrow})\psi_{\uparrow} + \psi^{\dagger}_{\downarrow} \vec{\nabla} \psi_{\downarrow} - (\vec{\nabla}\psi^{\dagger}_{\downarrow})\psi_{\downarrow}] \qquad (3.19)$$

Introducing the dynamical map (2.10) and taking only the

terms that contribute in the pair approximation we obtain

$$\vec{j}(x) = -\frac{1}{m}\int\frac{d^3k d^3\ell}{(2\pi)^6} e^{i\vec{x}\cdot\vec{\ell}}\vec{k}\sin(\theta_{k+\frac{1}{2}\ell} - \theta_{k-\frac{1}{2}\ell})\bar{\alpha}_{-k+\frac{1}{2}\ell\uparrow}(t)\bar{\alpha}_{k+\frac{1}{2}\ell\downarrow}(t)+h.c. \tag{3.20}$$

Next introducing the explicit expressions of $\bar{\alpha}$ and $\bar{\alpha}^\dagger$ in terms of $\alpha,\alpha^\dagger$ and considering the quasi-fermion contribution in the pair channel we have

$$\vec{j}^F = \vec{j}^{(0)} + \vec{j}^{(1)} \tag{3.21}$$

where $\vec{j}^{(0)}$ is defined by (3.11) and $\vec{j}^{(1)}$ is given by

$$\vec{j}^{(1)}(x) = \int\frac{d^3\ell}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{\ell}}\vec{J}(\vec{p},\vec{\ell})\alpha_{\ell-p\uparrow}(t)\alpha_{p\downarrow}(t)+h.c. \tag{3.22}$$

$$\vec{J}(\vec{p},\vec{\ell}) = \frac{1}{m}\int\frac{d^3k}{(2\pi)^3}\vec{k}\,\sin(\theta_{k+\frac{1}{2}\ell} - \theta_{k-\frac{1}{2}\ell})[f^{(1)}_{k+\frac{1}{2}\ell}(p,\ell) - f^{(2)}_{k+\frac{1}{2}\ell}(p,\ell)] \tag{3.23}$$

By means of (2.23) and (2.24) we can express $\vec{J}(\vec{p},\vec{\ell})$ in terms of the B-S amplitudes:

$$\vec{J}(\vec{p},\vec{\ell}) = \frac{\vec{\ell}}{\ell^2}\int\frac{d^3k}{(2\pi)^3}\,\frac{\vec{k}\cdot\vec{\ell}}{m}\,\frac{\lambda\sin(\theta_+-\theta_-)}{(E_++E_-)^2-E^2}\{(E_++E_-)\cos(\theta_+-\theta_-)[F^{(1)}(p,\ell) \tag{3.24}$$

$$+ F^{(2)}(p,\ell)] + E\cos(\theta_++\theta_-)[F^{(1)}(p,\ell) - F^{(2)}(p,\ell)] - E\sin(\theta_++\theta_-)f(p,\ell)\}$$

where by obvious notation $E_\pm = E_{k\pm\frac{1}{2}\ell}$, $\theta_\pm = \theta_{k\pm\frac{1}{2}\ell}$ and $E = E_p+E_{p-\ell}$. Calculating $\vec{J}$ at first order in ℓ and by means of (B.16) and (3.8) we have

$$\vec{J}(\vec{p},\vec{\ell}) = 2\Delta R(\vec{\ell},E)\,[F^{(1)}(p,\ell)+F^{(2)}(p,\ell)]\,\frac{\bar{\omega}_\ell^2}{\ell^2}\vec{\ell} \tag{3.25}$$

where the definition (2.41) has been used. From the B-S equations (2.36) we have

$$(1-Q^+-Q^{33})(F^{(1)}+F^{(2)})-EQ^-(F^{(1)}-F^{(2)})+2EL^-[1+q(\vec{\ell})]f = \cos(\theta_p-\theta_{p-\ell}); \tag{3.26}$$

neglecting the term proportional to $F^{(1)}-F^{(2)}$ since it is of higher order in ℓ than the present order of calculation and using the explicit expression (3.8) for $f(\vec{p},\vec{\ell})$ we find

$$F^{(1)}+F^{(2)} = \frac{1}{R[E^2-\omega_p(\vec{\ell})^2]}\,\frac{\omega_p(\vec{\ell})^2}{\bar{\omega}_\ell^2}\cos(\theta_p-\theta_{p-\ell}) + 2\Delta ER[1+q(\vec{\ell})]\sin(\theta_p+\theta_{p-\ell}) \tag{3.27}$$

where Eq. (2.39) and definition (3.7) have been used. From (3.25) and (3.27) we obtain for the Fourier transform of $\vec{j}^{(1)}$:

$$\vec{j}^{(1)}(\vec{\ell},t) = \int\frac{d^3p}{(2\pi)^3}\,\vec{J}(\vec{p},\vec{\ell})\,\alpha_{\ell-p\uparrow}(t)\,\alpha_{p\downarrow}(t)+\text{h.c.}$$

$$= \frac{1}{\frac{\partial^2}{\partial t^2}+\omega_p(\vec{\ell})^2}\left[i\vec{\ell}\,\frac{\bar{\omega}_\ell^2}{\ell^2}\frac{\partial}{\partial t}\,\rho^{(0)}(\vec{\ell},t)-\frac{\vec{\ell}}{\ell^2}\,\omega_p(\vec{\ell})^2\vec{\ell}\cdot\vec{j}^{(0)}(\vec{\ell},t)\right] \tag{3.28}$$

where $\vec{j}^{(0)}$ and $\rho^{(0)}$ are defined by (3.11) and (3.12), respectively. Substituting (3.28) in (3.21) we get the following expression for the quasi-fermion contribution to the current density:

$$\vec{j}^F(\vec{\ell},t) = \frac{1}{\frac{\partial^2}{\partial t^2} + \omega_p(\vec{\ell})^2}\left[\left(\frac{\partial^2}{\partial t^2} + \bar{\omega}_\ell^2\right)\vec{j}^{(0)} + i\vec{\ell}\,\frac{\bar{\omega}_\ell^2}{\ell^2}\partial_\mu j_\mu^{(0)}\right] \tag{3.29}$$

By noting that at the present order of computation[15]

$$\bar{\omega}_\ell = v_o \ell \tag{3.30}$$

where $v_o = v_F/\sqrt{3}$, and that

$$4\Delta^2 Rq(\vec{\ell})\,\bar{\omega}_\ell^2 = \frac{4\pi e^2 n_s}{m} = \omega_p^2 \tag{3.31}$$

where ω_p is the plasma frequency, we can rewrite the charge and current density in the coordinate space as

$$j_\mu^F(x) = \frac{\tilde{\Box}\,\delta_{\mu\nu} - \partial_\mu\partial_\nu}{\tilde{\Box} + \omega_p^2/v_o^2}\, j_\nu^{(0)} \tag{3.32}$$

where the symbol $\tilde{\Box}$ stands for the "D' Alambertain operator"

$$\tilde{\Box} = \frac{1}{v_o^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \tag{3.33}$$

with a velocity of propagation equal to v_o. The conservation of the fermion current

$$\partial_\mu j_\mu^F(x) = 0 \tag{3.34}$$

is immediately verified from the expression (3.32).

4) Calculation of the Boson Current

In this section we shall compute the contribution of the boson field to the charge and current density. In order to obtain a solution for the dynamical map (2.10)-(2.11) we must solve the B-S equations (2.37). Let us start by noticing that this system of linear equations will admit a non-trivial solution only when

$$W(\vec{\ell},\omega_\ell) = 0 \tag{4.1}$$

where W is given by (3.6) and ω_ℓ is the energy of the boson field B. The equation (4.1) will fix the energy spectrum of B; from (3.6) we find

$$\omega_\ell^2 = \begin{cases} 4\Lambda^2+\bar{\omega}_\ell^2 & (4.2) \\ \omega_p(\vec{\ell})^2 & (4.3) \end{cases}$$

It should be noted that these expressions are valid neglecting terms of the order of $0(\ell^4)$. Therefore we find that the dynamical map admits a solution corresponding to the presence of two collective modes among the free fields. The presence of a collective mode whose energy spectrum is given by (4.2) has not been reported in preceding studies (Cfr. Refs. (1) and (3)). We have not examined in detail this solution and we defer to a future article the problem of studying if this collective mode will contribute to dynamical effects. Only we would like to comment that this field does not contribute to the generator of the gauge transformation, so that its presence does not affect the boson method as described in

Ref. (4). In the present article we shall limit ourselves to consider only the boson field with an energy spectrum given by (4.3). According to (3.7) and (2.35) and by noticing that $4\Delta^2 R(0,0) = \lambda N(0)$, where $N(0)$ is the density of states at the Fermi level, ω_ℓ can be expressed as

$$\omega_\ell^2 = v_o^2 \ell^2 [1+\lambda N(0)] + \omega_p^2 - \frac{\varepsilon^2 \omega_p^2}{\ell^2+\varepsilon^2} \qquad (4.4)$$

At first let us note that in the limit $e \to 0$ (i.e. no Coulomb potential is present) we find a collective mode with a sound velocity

$$v_s = v_o [1+\lambda N(0)]^{1/2} \qquad (4.5)$$

already found by several authors[1,2,14]. It is worth noticing that this expression agrees with the result found in Ref. (14), but does not agree with the result of Anderson[1] which had a factor of 4 multiplying the second term in the bracket.

In the charged case special attention must be devoted to the quantity ε which, roughly speaking, corresponds to the inverse of the linear dimension of the system, $\varepsilon \simeq 1/L$. Keeping ε as an extremely small but finite quantity corresponds to the case when the system is not really infinite. From (4.4) we obtain:

$$\omega_\ell^2 = \begin{cases} \omega_p^2 + v_s^2 \ell^2 & \text{for } \ell >> \varepsilon \qquad (4.6) \\ v_s^2 \ell^2 & \text{for } \ell << \varepsilon \qquad (4.7) \end{cases}$$

This form for the energy spectrum of the boson field, already

reported in Ref. 3, is in contrast to previous studies[1,14] which found that in the presence of the Coulomb potential the collective mode acquires a finite mass which coincides with the plasma value. From (4.4) we find that practically the energy of the collective mode starts at the plasma frequency, but for the presence of (4.7) it is possible to have condensation of bosons in the ground state. This result furnishes a proof of the fact that the Goldstone theorem may be true even in the presence of long range interactions. In section 5 we shall discuss the Goldstone theorem and the rearrangement of the symmetry in the light of the results derived.

Let us now introduce the explicit expression for the boson field in terms of annihilation and creation operators:

$$B(x) = \Sigma_{\ell} \frac{1}{v_s \ell} \sqrt{\frac{\omega_\ell}{2}} \left[B_\ell e^{i\vec{\ell}\cdot\vec{x} - i\omega_\ell t} + B_\ell^\dagger e^{-i\vec{\ell}\cdot\vec{x} + i\omega_\ell t} \right] \tag{4.8}$$

This expression reduces to the usual expression free field when $e \to 0$. The field B will satisfy the equation of motion

$$\left[\frac{\partial^2}{\partial t^2} - v_s^2 \vec{\nabla}^2 + \omega_p^2 - \frac{\varepsilon^2 \omega_p^2}{\varepsilon^2 - \vec{\nabla}^2} \right] B(\vec{x},t) = 0 \tag{4.9}$$

which can be rewritten in the form:

$$\left[\frac{\partial^2}{\partial t^2} - v_s^2 \vec{\nabla}^2 \right] B(\vec{x},t) - \frac{\omega_p^2}{4\pi} \int d^3y \, \frac{1}{|\vec{x}-\vec{y}|} \, e^{-\varepsilon|\vec{x}-\vec{y}|} \vec{\nabla}^2 B(\vec{y},t) = 0 \tag{4.10}$$

The last term in (4.10) represents the contribution due to the Coulomb effect. We shall also introduce the canonical conjugate

of B, π:

$$\pi(x) = -i\Sigma_{\ell} \frac{v_s \ell}{\sqrt{2\omega_\ell}} \left[B_\ell e^{i\vec{\ell}\cdot\vec{x} - i\omega_\ell t} - B_\ell^\dagger e^{-i\vec{\ell}\cdot\vec{x} + i\omega_\ell t} \right] \tag{4.11}$$

It should be noted[3] that π is related to B by:

$$\pi = \dot{B} \quad \text{in the neutral case} \tag{4.12}$$

$$\pi(\vec{x},t) = -\frac{1}{4\pi}\int d^3y \; \frac{e^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \vec{\nabla}^2 \dot{B}(\vec{y},t) \quad \text{in the charged case} \tag{4.13}$$

where $\mu = \omega_p / v_s$.

In order to solve the B-S equations (2.32) we shall fix the normalization of the amplitude (2.28) by requiring[2] that the field B satisfies canonical commutation relations. We thus find the normalization condition:

$$\int \frac{d^3k}{(2\pi)^3} \{ |g_k^{(1)}(\ell)|^2 - |g_k^{(2)}(\ell)|^2 \} = 1 \tag{4.14}$$

where the functions $g^{(1)}$ and $g^{(2)}$ are defined by (2.25) and (2.26), respectively.

a) Calculation of the Charge Density.

From the definitions (2.22), (2.28) and (2.30) we have

$$\langle 0 | \rho(x) | B_\ell \rangle = g(x;\ell) \tag{4.15}$$

so that the boson contribution to the charge density is given by

$$\rho^B(x) = \int \frac{d^3\ell}{(2\pi)^3} e^{i\vec{x}\cdot\vec{\ell} - i\omega_\ell t} g(\ell) B_\ell + \text{h.c.} \tag{4.16}$$

Solving the equations (2.37) with the condition (4.14) we find

$$g(\ell) = -i2\Delta v_s \ell \sqrt{\frac{R(\ell)}{\lambda \omega_\ell}} \tag{4.17}$$

Substituting this result in (4.16) and taking into account the definition (4.11) we obtain

$$\rho^B(\vec{x},t) = -\eta(\partial)\pi(\vec{x},t) \tag{4.18}$$

where the coefficient $\eta(\partial)$ is defined by

$$\eta(\partial) = -2\Delta\sqrt{\frac{2}{\lambda}R(\partial)} \tag{4.19}$$

b) Calculation of the Current Density.

Substituting the dynamical map (2.10)-(2.11) in (3.19) and considering the boson contribution in the pair approximation:

$$\vec{j}^B(\vec{x},t) = \frac{1}{m}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3\ell}{(2\pi)^3}e^{i\vec{x}\cdot\vec{\ell}-i\omega_\ell t}\vec{k}\sin(\theta_{k+\frac{1}{2}\ell}-\theta_{k-\frac{1}{2}\ell})$$

$$+ [g^{(1)}_{k+\frac{1}{2}\ell}(\ell)-g^{(2)}_{k+\frac{1}{2}\ell}(\ell)]B_\ell+h.c. \tag{4.20}$$

From (4.17) and from the B-S equations we can evaluate the functions $g^{(1)}$ and $g^{(2)}$. The final result is

$$\vec{j}^B(\vec{x},t) = v_s^2\eta(\partial)\vec{\nabla}B(\vec{x},t) \tag{4.21}$$

where use has been made of definition (4.8). By noticing that from (4.9) the boson field satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2} - v_s^2 \vec{\nabla}^2 + \omega_p^2\right] \vec{\nabla}^2 B(\vec{x},t) \qquad (4.22)$$

we obtain from this relation and from (4.13) the important result:

$$\dot{\pi}(\vec{x},t) = v_s^2 \vec{\nabla}^2 B(\vec{x},t) \qquad (4.23)$$

Owing to this relation the conservation of the current is immediately verified:

$$\partial_\mu j_\mu^B = \vec{\nabla}\cdot\vec{j}^B + \frac{\partial}{\partial t}\rho^B = \eta(\partial)[v_s^2 \vec{\nabla}^2 B - \dot{\pi}] = 0 \qquad (4.24)$$

As we have already mentioned Eq. (4.22) is very different from the equivalent equations obtained in other approaches. The important point in (4.22) is that this equation admits the solution

$$\vec{\nabla}^2 B = 0 \qquad (4.25)$$

The existence of this solution will permit boson condensation in the ground state which recovers the gauge invariance of the theory.

5) Long Range Correlations, Goldstone theorem and Rearrangement of Symmetries in Superconductivity.

The conditions of applicability of the Goldstone theorem can be schematized in the following three points:

a) a Lagrangian theory which has an internal symmetry group;

b) asymmetric nature of the vacuum (i.e. conditions of long range order);

c) the generator of the symmetry group is time independent[16].

The last condition has brought a series of interesting speculations concerning the study of the cases in which the Goldstone theorem appears to be invalidated. In particular, since the condition c) is always satisfied in the case of manifestly covariant theories, attention has been devoted to non relativistic theories and to relativistic theories which do not present manifest covariance. The experimental evidence that long range excitations are present in superfluid liquids, but absent in superconducting systems, and the theoretical work of Anderson,[1] brought people to speculate about the possibility that the Goldstone theorem breaks down when long range fields are present in the theory. This circumstance has special interest in the physics of elementary particles if one wants to explain the presence of massive observed particles without appealing to the introduction of breaking terms in the Lagrangian.

The common argument that relates the presence of long range fields to the invalidation of the Goldstone theorem is the failure of the condition c). If j_μ is the conserved current we have from the conservation law

$$\dot{N} \equiv \frac{\partial}{\partial t} \int d^3x j_o(x) = -\int \vec{ds} \cdot \vec{j}(x) \qquad (5.1)$$

The generator N will be time independent if and only if the surface contribution of the current vanishes. In non relativist

theories the presence of long range forces may imply[18] that there exists a flow of current passing through the surface which contains the system; as a consequence in (5.1) we shall have a contribution from the surface integral and the generator will be time dependent. In relativistic theories which are not manifestly covariant the requirement of causality does not necessarily apply, so that one might expect a non vanishing contribution from the spatial integral in (5.1)[19].

We would like to observe that this argument does not bring much information since no general statement can be reached about the effects of the original invariance: the problem must be solved case by case according to the dynamics of the particular system under study. Also this scheme of thinking may bring serious difficulties in quantum field theory since the existence of incoming fields is not necessarily justified in the case in which effects from infinity occur. As we have seen our results contain surface effects only through ε which fixes the shape of the energy spectrum around $\ell = 0$; we will show that the generator is time independent, and does not depend on these effects. This circumstance preserves the original invariance of the theory.

In relativistic theories the effects of long range fields on the Goldstone theorem have been studied in different models. The Goldstone model[20] interacting with a gauge field has been considered by Higgs[11] in a semi-classical treatment, and by Englert and Brout[11] in the lowest order perturbation theory.

The Nambu and Jona-Lasinio model[21] with an axial vector gauge field coupled to the helicity current has been studied by Freundlich and Lurié[12]. In all the models considered the results turn out to coincide with those reached by Anderson in superconductivity. The introduction of the gauge field eliminates completely the Goldstone particle (present in the original Golstone and Nambu and Jona-Lasinio models) while at the same time the gauge field acquires a finite mass.

All these studies in relativistic and non relativistic theories seem to show that the Goldstone theorem is not valid anymore when long range interactions among the particles are present at the level of the Heisenberg fields. However, a more careful analysis shows that all these considerations are not completely satisfactory and many problems remain unanswered. We have already mentioned that the original invariance of the theory must be conserved even at the stage of the physical particles; spontaneous breakdown of symmetry implies in reality that the form of the invariance transformation takes a different shape when written in terms of the physical fields. If we examine all the works previously mentioned it is easy to see that the original invariance transformations vanish entirely from the theory when expressed in terms of physical fields; in other words the physical fields do not contribute to the generator and remain completely unchanged under the transformation. This conclusion is rather surprising since we should expect that at the level of the physical fields some entity must carry the transformation. This aspect is closely related to the fact that apparently

the Goldstone particle has completely disappeared from the theory without leaving any trace. When the rearrangement of the symmetry has been studied in different models using the self-consistent method,[13] it has always been found that at the level of the physical fields the Goldstone particles are the only fields which contribute to the generator and which transport all the symmetry attributes.

Summarizing, we would like to say that the problem of spontaneous breakdown of symmetries when long range correlations are present has not been studied with sufficient care, and more analysis is required. One of the motivations of the present work has been to study this problem in non relativistic theories taking in consideration the BCS model in superconductivity. By using the self-consistent method we have solved the model in the pair approximation and an answer can be given to the questions asked before.

Let us consider the constant gauge transformation of the electron field

$$\psi \longrightarrow \psi' = e^{-iN\theta}\psi e^{iN\theta} = e^{i\theta}\psi \qquad (5.2)$$

where

$$N = \int d^3x \rho(x); \qquad (5.3)$$

θ is a constant real parameter and ρ is defined by (2.6). The transformation (5.2) leaves the Hamiltonian (2.1) invariant. Let us now examine how the physical fields $\Phi_{\uparrow,\downarrow}$ and B change under the transformation induced by (5.3). Taking into account the results (3.18) and (4.18) the generator (5.3) can be

expressed in terms of physical fields and takes the form

$$N = -\eta(\partial)\int d^3x\pi(x) \tag{5.4}$$

The first consequence of this expression is that the quasi-fermion fields $\Phi_{\uparrow,\downarrow}$ do not contribute to the generator and remains invariant under the gauge transformation:

$$\Phi_{\uparrow,\downarrow} \longrightarrow \Phi_{\uparrow,\downarrow} \tag{5.5}$$

while the boson field transforms as

$$B(x) \longrightarrow B'(x) = e^{-iN\theta}B(x)e^{iN\theta} = B(x) + \eta\theta \tag{5.6}$$

$$\pi \longrightarrow \pi \tag{5.7}$$

where $\eta = \eta(0)$. The second important consequence which follows from (5.4) is that the generator is time independent

$$\dot{N} = 0 \tag{5.8}$$

This result is immediately proved by recalling the expressions (4.11) and (4.4). The conclusion (5.8) can be realized without introducing Goldstone excited levels by means of the presence of the Laplace operator in Eq. (4.22); owing to the presence of $\vec{\nabla}^2$ the system exhibits a condensation in the ground state and the transformation (5.6) leaves invariant the equations of motion (4.9) and (4.10) for the boson field. We thus find that the gauge transformation (5.2) of the electron field is re-arranged at the level of the free fields in the shape of a linear transformation of the boson field; the original gauge invariance of the theory is not lost but is recovered by the presence of

the boson field which acts as a Goldstone particle. From (5.5)-(5.7) it also follows that the electron fields $\psi_{\uparrow,\downarrow}$, expressed in terms of physical fields, take the form

$$\psi_{\uparrow,\downarrow} = e^{i\frac{1}{\eta}B} F(\phi, \vec{\nabla}B, \pi) \tag{5.9}$$

where F is a certain unknown function.

In conclusion, we would like to state that our results, in contrast with previous studies[1], seem to indicate that the Goldstone theorem is still valid in the presence of long range forces in the sense that a boson condensation is still permitted. A collective mode whose energy goes to zero in the limit of of long wave lengths is present in the theory; this massless particle is not only responsible for the asymmetry of the ground state, but rearranges and preserves the symmetry of the theory when formulated in terms of quasi-fields.

Similar conclusions have been recently reached[22] in relativistic theories in the study of non convariant solutions[23] of the generalized Nambu and Jona-Lasinio model[11].

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Appendix A

The kernels Q^{ij} which appear in the Bethe-Salpeter equations are defined by:

$$Q^{11} = Q^{+} + EQ^{-} \qquad Q^{22} = Q^{+} - EQ^{-}$$

$$Q^{+}(\vec{\ell},E) = -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}(E_kE_{k-\ell}+\varepsilon_k\varepsilon_{k-\ell})(E_k+E_{k-\ell})f(\vec{k},\vec{\ell},E)$$

$$Q^{-}(\vec{\ell},E) = -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}(\varepsilon_kE_{k-\ell}+\varepsilon_{k-\ell}E_k)f(\vec{k},\vec{\ell},E)$$

$$Q^{13} = EL^{-} + L^{+} \qquad Q^{23} = EL^{-} - L^{+}$$

$$L^{+}(\vec{\ell};E) \ -\frac{\Delta}{4}\int\frac{d^3k}{(2\pi)^3}(\varepsilon_k+\varepsilon_{k-\ell})(E_k+E_{k-\ell})f(\vec{k},\vec{\ell},E)$$

$$L^{-}(\vec{\ell},E) = -\Delta R(\vec{\ell},E) \qquad Q^{33}(\vec{\ell},E) = -2\Delta^2 R(\vec{\ell},E)$$

$$Q^{12}(\vec{\ell},E) = -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}(E_kE_{k-\ell}-\varepsilon_k\varepsilon_{k-\ell})(E_k+E_{k-\ell})f(\vec{k},\vec{\ell},E)$$

where $R(\vec{\ell},E)$ is defined by (2.36) and

$$f(\vec{k},\vec{\ell},E) = \frac{\lambda}{E_kE_{k-\ell}}\ \frac{1}{[(E_k+E_{k-\ell})^2-\omega_\ell^2]}$$

Appendix B

The quantities K, X_2, X_3, Y, Y', Z, Z', Z_2 introduced in section 3 are defined in terms of the coefficients Q^{ij} in Appendix A. Their explicit expressions are

$$K = [RY+q(\vec{\ell})Y'][RY-Z+q(\vec{\ell})(Y'-Z')]^{-1} \tag{B.1}$$

$$X_2 = L^{+}(1-Q^{+}-Q^{33})+E^2L^{-}Q^{-} \tag{B.2}$$

$$X_3 = (1-Q^{+}-Q^{33})(1-Q^{+}+Q^{33})-E^2Q^{-2} \tag{B.3}$$

$$Y = (1-Q^{+}+Q^{33})(1-Q^{12}-Q^{33})-4L^{+2} \tag{B.4}$$

$$Y' = -R(1-Q^{+}+Q^{33})(Q^{12}+Q^{33})-4RL^{+2} \tag{B.5}$$

$$Z_1 = Q^{-}(1-Q^{12}-Q^{33})+4L^{+}L^{-} \tag{B.6}$$

$$Z_2 = L^{-}(1-Q^{+}+Q^{33})+L^{+}Q^{-} \tag{B.7}$$

$$Z = Q^{-}Z_1+4L^{-}Z_2 \tag{B.8}$$

$$Z' = Z-Q^{-2} \tag{B.9}$$

For values of ℓ sufficiently small with respect to k_D, the Debye momentum, the domain of integration in Eq. (2.34) can be considered ℓ-independent; using the relation (2.35) and the expressions of the coefficients Q^{ij} the following results can be obtained:

$$X_2(\vec{\ell},E) \simeq 0(\ell^4) \tag{B.10}$$

$$X_3(\vec{\ell},E) = R^2(E^2-\bar{\omega}_\ell^2)(E^2-\bar{\omega}_\ell^2-4\Delta^2)+0(\ell^4) \qquad (B.11)$$

$$Y(\vec{\ell},E) = R(1+4\Delta^2R)(E^2-\bar{\omega}_\ell^2-4\Delta^2)+0(\ell^4) \qquad (B.12)$$

$$Z(\vec{\ell},E) = Z'(\vec{\ell},E) = Y'(\vec{\ell},E) = 4\Delta^2R^3(E^2-\bar{\omega}_\ell^2-4\Delta^2)+0(\ell^4) \qquad (B.13)$$

$$Z_2(\vec{\ell},E) = -\Delta R^2(E^2-\bar{\omega}_\ell^2-4\Delta^2)+0(\ell^4) \qquad (B.14)$$

Substituting these values in the expression of K we obtain

$$K(\vec{\ell},E) = 1+4\Delta^2R(\vec{\ell},E)+4\Delta^2R(\vec{\ell},E)q(\vec{\ell})+0(\ell^2) \qquad (B.15)$$

The result (B.10) is derived observing that

$$\Delta Q^- - L^+ = \frac{\lambda\Delta}{4}\int\frac{d^3k}{(2\pi)^3}\frac{\vec{k}\cdot\vec{\ell}}{m}\frac{E_k-E_{k-\ell}}{E_kE_{k-\ell}[(E_k+E_{k-\ell})^2-E^2]} \simeq 0(\ell^4) \qquad (B.16)$$

References

1) P. W. Anderson, Phys. Rev. 110, 827 and 112, 1900 (1958); N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, A New Method in the Theory of Superconductivity, Academy of Science, Moscow 1958, Consultants Bureau, New York 1959; Y. Nambu, Phys. Rev. 117, 648 (1960).

2) L. Leplae, and H. Umezawa, Nuovo Cimento 44, 410 (1966).

3) L. Leplae, and H. Umezawa, Journ. Math. Phys. 10, 2038 (1969).

4) L. Leplae, F. Mancini, and H. Umezawa, Phys. Rev. B 2, 3594 (1970)
L. Leplae, F. Mancini, and H. Umezawa, Phys. Rev. to be published.

5) L. Leplae, F. Mancini, and H. Umezawa, Lettere al Nuovo Cimento 3, 153 (1970); see also the first paper in Ref. 4

6) L. Leplae, F. Mancini, and H. Umezawa, Lettere al Nuovo Cimento 4, 963 (1970); Nuovo Cimento, to be published; Physics Letters 34A, 301 (1971); Preprint UWM-4867-71-9.

7) L. P. Gor'kov, and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 54, 612 (1968) [Sov. Phys. JETP 27, 328 (1968)].

8) J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

9) L. Leplae, R. N. Sen, and H. Umezawa, Progr. Theor. Phys. Suppl., Extra Number, 637 (1965).

H. Umezawa, Nuovo Cimento 40, 450 (1965)

K. Nakagawa, R. N. Sen, and H. Umezawa, Nuovo Cimento 42, 565 (1966)

L. Leplae, R. N. Sen, and H. Umezawa, Nuovo Cimento 49, 1 (1967)

R. N. Sen, and H. Umezawa, Nuovo Cimento 50, 53 (1967).

P. W. Higgs, Phys. Letters 12, 132 (1964); Phys. Rev. 145, 1156 (1966)

F. Englert, and R. Brout, Phys. Rev. Letters 13, 321 (1964

T. W. Kibble, Phys. Rev. 155, 1554 (1967).

Y. Freundlich, and D. Lurié, Nucl. Phys. B19,557 (1970).

H. Umezawa, Acta Phys. Hung. 19, 9 (1965).

V. Ambegaokar, and L. P. Kadanoff, Nuovo Cimento 22, 914 (1961).

It can be shown that the approximation used can be applied for all range of momenta with $\ell < 1/\xi$.

It is well known[17] that in any case of spontaneous breakdown of symmetry the generator is not defined as an operator acting on the physical representation. This circumstance does not involve any difficulty since in all the practical calculations one is concerned only with commutators of N with other generators; the generator does not need to be defined, what is important is the algebra that it satisfies. However, we can still use, at least in a formal sense, the notion of a generator if we adopt the following definition. Let $\{g_n\}$ be a sequence of

square integrable functions uniformly convergent to 1 for $n \to \infty$. Then let us introduce the generator

$$N_{g_n} = \int d^3x g_n(x) j_o(x)$$

which is always defined. In all the cases in which we shall use the generator this must be intended as $N = \lim_{n\to\infty} N_{g_n}$, where the limit $n \to \infty$ must be taken once all the calculations have been performed.

17) E. Fabri, and L. E. Picasso, Phys. Rev. Letters 16, 408 (1966).

18) R. Lange, Phys. Rev. Letters 14, 3 (1965).

19) G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Letters 13, 585 (1964).

20) J. Goldstone, Nuovo Cimento 19, 154 (1961).

21) Y. Nambu, and G. Jona-Lasinio, Phys. Rev. 122, 345; 124, 246 (1961).

22) A. Aurilia, Y. Takahashi, and H. Umezawa, Preprint, University of Alberta, Canada (1971).

23) W. Gilbert, Phys. Rev. Letters 17, 713 (1964).